FREE FIELD REALIZATION OF VERTEX OPERATORS FOR LEVEL TWO MODULES OF $U_q\big(\widehat{\mathfrak{sl}}(2)\big)$

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Abstract. Free field realization of vertex operators for level two modules of $U_q\big(\widehat{\mathfrak{sl}}(2)\big)$ are shown through the free field realization of the modules given by Idzumi in Ref.[4, 5]. We constructed types I and II vertex operators when the spin of the associated evaluation module is 1/2 and type II's for the spin 1.

1. Introduction

Vertex operators for the quantum affine algebra $U_q\big(\widehat{\mathfrak{sl}}(2)\big)$ have played essential roles in the algebraic analysis of solvable lattice models since the pioneering works of [1, 2, 3]. In these works which analyze the XXZ model, type I vertex operators are identified with half infinite transfer matrices as their representation-theoretical counter part and type II vertex operators are interpreted as particle creation operators. To perform concrete computation such as a trace of composition of vertex operators, we need free field realization of modules and operators. In the said example of the XXZ model, the integral expressions of n-point correlation functions which are special cases of the traces are obtained through bosonization of level one module of $U_q\big(\widehat{\mathfrak{sl}}(2)\big)$.

Motivated by these results, Idzumi [4, 5] constructed level two modules and type I vertex operators accompanied by spin 1 evaluation modules for $U_q\big(\widehat{\mathfrak{sl}}(2)\big)$ in terms of bosons and fermions and then calculated correlation functions of a spin 1 analogue of the XXZ model. The purpose of this paper is to extend Idzumi's free field realization to other kinds of vertex operators i.e. type I and II vertex operators for the level two modules associated with the evalution module of spin 1/2 and the type II's for the spin 1. The results are given in Section 3 and their derivation is discussed in the first case in Section 4. The results together with Ref.[4, 5] give the complete set of vertex operators for level two module of $U_q\big(\widehat{\mathfrak{sl}}(2)\big)$ and enable one to calculate form factors of the spin 1 analogue of the XXZ model.

Recently Jimbo and Shiraishi [7] showed a coset-type construction for the deformed Virasoro algebra with the vertex operators for $U_q\big(\widehat{\mathfrak{sl}}(2)\big)$. They constructed a primary operator for the deformed Virasoro algebra as coset type composition of vertex operators which may be denoted as $\Big(U_q\big(\widehat{\mathfrak{sl}}(2)\big)_k \oplus U_q\big(\widehat{\mathfrak{sl}}(2)\big)_1\Big)/U_q\big(\widehat{\mathfrak{sl}}(2)\big)_{k+1}$. We hope that our results will be helpful

for extending this work to the deformed supersymmetric Virasoro algebra through $\Big(U_q\big(\widehat{\mathfrak{sl}}(2)\big)_k \oplus U_q\big(\widehat{\mathfrak{sl}}(2)\big)_2\Big)/U_q\big(\widehat{\mathfrak{sl}}(2)\big)_{k+2}$.

2. Free field realization of level two module

2.1. **Convention.** In the following we will use U to denote the quantum affine algebra $U_q\big(\widehat{\mathfrak{sl}}(2)\big)$. Unless mentioned, we follow the notations of Ref.[4, 5]. As for the free field representation, we slightly modify the convention.

The quantum affine algebra U is an associative algebra with unit 1 generated by e_i, f_i $(i = 0, 1), q^h$ $(h \in P^*)$ with relations

$$q^0 = 1,\ q^h q^{h'} = q^{h+h'},$$

$$q^h e_i q^{-h} = q^{\langle h,\alpha_i\rangle} e_i,\ q^h f_i q^{-h} = q^{-\langle h,\alpha_i\rangle} f_i,$$

$$[e_i, f_i]^* = \delta_{ij}\frac{t_i - t_i^{-1}}{q - q^{-1}},\ (t_i = q^{h_i})$$

$$e_i^3 e_j - [3]e_i^2 e_j e_i + [3]e_i e_j e_i^2 - e_j e_i^3 = 0,$$

$$f_i^3 f_j - [3]f_i^2 f_j f_i + [3]f_i f_j f_i^2 - f_j f_i^3 = 0,$$

where $P = \mathbb{Z}\Lambda_0 + \mathbb{Z}\Lambda_1 + \mathbb{Z}\delta$ is the weight lattice of the affine Lie algebra $\widehat{\mathfrak{sl}}(2)$ and P^* is the dual lattice to P with the dual basis $\{h_0, h_1, d\}$ to $\{\Lambda_0, \Lambda_1, \delta\}$ with respect to the natural pairing $\langle\ ,\ \rangle : P \times P^* \to \mathbb{Z}$. We also use current type generators introduced by Drinfeld [11]

$$[a_k.a_l] = \delta_{k+l,0}\frac{[2k]}{k}\frac{\gamma^k - \gamma^{-k}}{q - q^{-1}},$$

$$K a_k K^{-1} = a_k,\ \ K x_k^{\pm} K^{-1} = q^{\pm 2} x_k^{\pm},$$

$$[a_k, x_l^{\pm}] = \pm\frac{[2k]}{k}\gamma^{\mp|k|/2} x_{k+l}^{\pm},$$

$$x_{k+l}^{\pm} x_l^{\pm} - q^{\pm 2} x_l^{\pm} x_{k+l}^{\pm} = q^{\pm 2} x_k^{\pm} x_{l+1}^{\pm} - x_{l+1}^{\pm} x_k^{\pm},$$

$$[x_k^+, x_l^-] = \frac{\gamma^{\frac{k-l}{2}}\psi_{k+l} - \gamma^{\frac{l-k}{2}}\phi_{k+l}}{q - q^{-1}},$$

where ψ_k, and φ_k are defined as

$$\sum_{k\geqslant 0}\psi_k z^{-k} = K\exp\big\{(q - q^{-1})\sum_{k\geqslant 1} a_k z^{-k}\big\},$$

*[A,B]=AB-BA

$$\sum_{k\geqslant 0}\phi_k z^k = K^{-1}\exp\{-(q-q^{-1})\sum_{k\geqslant 1}a_{-k}z^k\}.$$

The relation between two types of generators are

$$t_1 = K,\ t_0 = \gamma K^{-1},\ e_1 = x_0^+,\ e_0 t_1 = x_1^-,\ f_1 = x_0^-,\ t_1^{-1}f_1 = x_0^{-1}.$$

The higest weight module and the evaluation module are described compactly in Ref.[4].

Commutation and anti commutation relations of bosons and fermions are given by

$$[a_m, a_n] = \delta_{m+n,0}\frac{[2m]^2}{m},$$

$$\{\phi_m, \phi_n\}^* = \delta_{m+n,0}\eta_m,$$

$$\eta_m = q^{2m} + q^{-2m}.$$

where $m, n \in \mathbb{Z}+1/2$ or $\in \mathbb{Z}$ for Neveu-Schwarz-sector or Ramond-sector respectively. Fock spaces and vacuum vectors are denoted as $\mathcal{F}^a,\ \mathcal{F}^{\phi^{NS}},\ \mathcal{F}^{\phi^R}$ and $|vac\rangle$,$|NS\rangle$, $|R\rangle$ for the boson and NS and R fermion respectively. Fermion currents are defined as

$$\phi^{NS}(z) = \sum_{n\in\mathbb{Z}+\frac{1}{2}}\phi_n^{NS}z^{-n},\ \phi^R(z) = \sum_{n\in\mathbb{Z}}\phi_n^R z^{-n}.$$

$Q = \mathbb{Z}\alpha$ is the root lattice of $\mathfrak{sl}_2$ and $F[Q]$ be the group algebra. We use ∂ as

$$[\partial, \alpha] = 2.$$

2.2. $V(2\Lambda_0), V(2\Lambda_1)$. The highest weight module $V(2\Lambda_0)$ is identified with the Fock space

$$\mathcal{F}_+^{(0)} = \mathcal{F}^a \otimes \{(\mathcal{F}_{even}^{\phi^{NS}} \otimes F[2Q]) \oplus (\mathcal{F}_{odd}^{\phi^{NS}} \otimes e^\alpha F[2Q])\}, \tag{1}$$

subscripts *even* and *odd* represent the number of fermions. The highest weight vector is $|vac\rangle \otimes |NS\rangle \otimes 1$. $V(2\Lambda_1)$ is

$$\mathcal{F}_-^{(0)} = \mathcal{F}^a \otimes \{(\mathcal{F}_{even}^{\phi^{NS}} \otimes e^\alpha F[2Q]) \oplus (\mathcal{F}_{odd}^{\phi^{NS}} \otimes F[2Q])\} \tag{2}$$

with the highest weight vector being $|vac\rangle \otimes |NS\rangle \otimes e^\alpha$. Note that

$$\mathcal{F}^{(0)} = \mathcal{F}_-^{(0)} \oplus \mathcal{F}_+^{(0)}.$$

$$\mathcal{F}^{(0)} = \mathcal{F}^a \otimes \mathcal{F}^{\phi^{NS}} \otimes F[Q].$$

The operators are realized in the following manner.

$$\gamma = q^2, \quad K = q^\partial,$$

$$x^\pm(z) = \sum_{m\in\mathbb{Z}} x_m^\pm z^{-m} = E_<^\pm(z)E_>^\pm(z)\phi^{NS}(z)e^{\pm\alpha}z^{\frac{1}{2}\pm\frac{1}{2}\partial},$$

$^*\{$A,B$\}$=AB-BA

$$E^{\pm}_{<}(z) = \exp\,(\pm \sum_{m>0} \frac{a_{-m}}{[2m]} q^{\mp m} z^{m}), \;\; E^{\pm}_{>}(z) = \exp\,(\mp \sum_{m>0} \frac{a_{m}}{[2m]} q^{\mp m} z^{-m}),$$

and

$$d = -\frac{\partial^2}{8} + \frac{(\lambda,\lambda)}{4} - \sum_{m=1}^{\infty} m N^a_m - \sum_{k>0} k N^{\phi^{NS}}_k, \tag{3}$$

$$N^a_m = \frac{m}{[2m]^2} a_{-m} a_m, \quad N^{\phi^{NS}}_k = \frac{1}{\eta_m} \phi^{NS}_{-m} \phi^{NS}_m \quad (m>0), \tag{4}$$

where the higest weight vector of the module should be substituted for λ of (3).

2.3. $V(\Lambda_0+\Lambda_1)$**.** The module $V(\Lambda_0+\Lambda_1)$ is identified with

$$\mathcal{F}^{(1)} = \mathcal{F}^a \otimes \mathcal{F}^{\phi^R} \otimes e^{\frac{\alpha}{2}} F[Q], \tag{5}$$

where

$$\phi^R_0 |R\rangle = |R\rangle.$$

The highest weight vector is identified with $|vac\rangle \otimes |R\rangle \otimes e^{\frac{\alpha}{2}}$.

Operators are constructed in the same way as before except that subscripts for fermion sector are R instead of NS.

3. Free field realizations of vertex operators

Let V, V' be level two modules and $V^{(k)}_z$ be a spin k/2 evaluation module of U. Vertex operators we will consider are U-linear maps of the following kinds [8, 9]

$$\Phi^{V',k}_V(z) : V \longrightarrow V' \otimes V^{(k)}_z, \tag{6}$$

$$\Psi^{k,V'}_V(z) : V \longrightarrow V^{(k)}_z \otimes V'. \tag{7}$$

Vertex operators of the form (6,7) are called type I and II respectively. Components of vertex operators are defined as

$$\Phi(z)^{V',k}_V = \sum_{n=0}^{k} \Phi_n(z) \otimes u_n, \;\; \Psi(z)^{k,V'}_V = \sum_{n=0}^{k} u_n \otimes \Psi_n(z).$$

3.1. **type I Vertex Operators for level 2 and spin 1/2.** We show free field realization of type I vertex operators of the following kinds

$$\Phi^{\Lambda_0+\Lambda_1,1}_{2\Lambda_i}(z) : V(2\Lambda_i) \longrightarrow V(\Lambda_0+\Lambda_1) \otimes V^{(1)}_z, \tag{8}$$

$$\Phi^{2\Lambda_i,1}_{\Lambda_0+\Lambda_1}(z) : V(\Lambda_0+\Lambda_1) \longrightarrow V(2\Lambda_i) \otimes V^{(1)}_z \tag{9}$$

where $i = 0$ or 1.

Under the free field realization of level 2 modules reviewed in Secton 2, the explicit forms of the components of the vertex operators in (8) are

$$\Phi_1(z) = B_{I,<}(z)B_{I,>}(z)\Omega^R_{NS}(z)e^{\alpha/2}(-q^4z)^{\partial/4}, \tag{10}$$

$$\begin{aligned}\Phi_0(z) = &\oint \frac{dw}{2\pi i} B_{I,<}(z)E^-_<(w)B_{I,>}(z)E^-_>(w)\Omega^R_{NS}(z)\phi^{NS}(w) \\ &\times e^{-\alpha/2}(-q^4z)^{\partial/4}w^{-\partial/2}(-q^4zw^3)^{-\frac{1}{2}}\frac{(\frac{w}{q^3z};q^4)_\infty}{(\frac{w}{qz};q^4)_\infty}\Big\{\frac{w}{1-q^{-3}w/z}+\frac{q^5z}{1-q^5z/w}\Big\},\end{aligned} \tag{11}$$

$$B_{I,<}(z) = \exp\big(\sum_{n=1}^\infty \frac{[n]a_{-n}}{[2n]^2}(q^5z)^n\big), \tag{12}$$

$$B_{I,>}(z) = \exp\big(-\sum_{n=1}^\infty \frac{[n]a_{n}}{[2n]^2}(q^3z)^{-n}\big). \tag{13}$$

The integrand of $\Phi_0(z)$ has poles only at $w = q^5z, q^3z$ except for $w = 0, \infty$ and the contour of integration encloses $w = 0, q^5z$, details are discussed in Sec.4. For those of (9) we just replace $\Omega^R_{NS}(z)$ with $\Omega^{NS}_R(z)$ in (10,11).

The fermionic part $\Omega(z)$'s are maps between different fermion sectors and satisfy

$$\phi^{NS}(w)\Omega(z)^{NS}_R = \Big(\frac{-q^4z}{w}\Big)^{1/2}\frac{(\frac{w}{q^3z};q^4)_\infty(\frac{q^7z}{w};q^4)_\infty}{(\frac{w}{qz};q^4)_\infty(\frac{q^5z}{w};q^4)_\infty}\Omega(z)^{NS}_R\phi^R(w) \tag{14}$$

and exactly the same equation except subscripts for fermion sectors are exchanged. This kind of mapping for fermions first appeared in high-energy phisics theory as "fermion emission vertex operator"[6, 10]. Their free field realizations are

$$\Omega^R_{NS}(z) = \langle NS|\, e^Y|R\rangle, \tag{15}$$

$$\begin{aligned}Y = &-\sum_{m>n\geq 0} X_{m,n}\varphi^R_{-m}\varphi^R_{-n}z^{m+n} - \sum_{k>l\geq 0} X_{k+1/2,l+1/2}\varphi^{NS}_{k+1/2}\varphi^{NS}_{l+1/2}z^{-k-l-1} \\ &+\sum_{\substack{m\geq 0\\ k\geq 0}} X_{m,-k-1/2}\varphi^R_{-m}\varphi^{NS}_{k+1/2}z^{m-k-1/2}\end{aligned} \tag{16}$$

$$\Omega_R^{NS}(z) = \langle R|\ e^{Y'}|NS\rangle, \tag{17}$$

$$\begin{aligned} Y' = &\sum_{k>l\geq 0} X_{k+1/2,l+1/2}\varphi^{NS}_{-k-1/2}\varphi^{NS}_{-l-1/2}z^{k+l+1} + \sum_{m>n\geq 0} X_{m,n}\varphi^R_m\varphi^R_n z^{-m-n} \\ &- \sum_{\substack{k\geq 0\\ m\geq 0}} X_{-k-1/2,m}\varphi^{NS}_{-k-1/2}\varphi^R_m z^{k-m+1/2} \end{aligned} \tag{18}$$

$$\varphi^R_0 = \phi^R_0, \quad \varphi^R_{-m} = \phi^R_{-m}\frac{\gamma_m q^{5m}}{\eta_m}, \quad \varphi^R_m = \phi^R_m \frac{\gamma_m q^{-3m}}{\eta_m} \quad (m>0), \tag{19}$$

$$\varphi^{NS}_{k+1/2} = \phi^{NS}_{k+1/2}\frac{\gamma_k q^{-3k-2}}{\eta_{k+1/2}}(-(-1)^{1/2}), \quad \varphi^{NS}_{-k-1/2} = \phi^{NS}_{-k-1/2}\frac{\gamma_k q^{5k+2}}{\eta_{k+1/2}}(-1)^{1/2} \quad (k>0), \tag{20}$$

$$X_{k,l} = \frac{q^{4k}-q^{4l}}{1-q^{4(k+l)}},$$

$$\gamma_n = \frac{(q^2;q^4)_n}{(q^4;q^4)_n}, \quad \frac{(q^2z;q^4)_\infty}{(z;q^4)_\infty} = \sum_{n=0}^\infty \gamma_n z^n. \tag{21}$$

(15,17) are to mean that a matrix element is given by

$$_R\langle \text{out}|\Omega^R_{NS}(z)|\text{in}\rangle_{NS} = {}_R\langle\text{out}| \otimes \langle NS|\ e^Y|R\rangle \otimes |\text{in}\rangle_{NS},$$
$$\text{for } |\text{out}\rangle_R \in \mathcal{F}^{\phi^R},\ |\text{in}\rangle_{NS} \in \mathcal{F}^{\phi^{NS}}.$$

We define the normalized vertex operators $\tilde{\Phi}(z)$'s as follows

$$\langle \Lambda_0+\Lambda_1|\tilde{\Phi}_1(z)|2\Lambda_0\rangle = 1, \quad \langle 2\Lambda_1|\tilde{\Phi}_1(z)|\Lambda_0+\Lambda_1\rangle = 1,$$
$$\langle \Lambda_0+\Lambda_1|\tilde{\Phi}_0(z)|2\Lambda_1\rangle = 1, \quad \langle 2\Lambda_0|\tilde{\Phi}_0(z)|\Lambda_0+\Lambda_1\rangle = 1,$$

and these are given by

$$\tilde{\Phi}^{\Lambda_0+\Lambda_1,1}_{2\Lambda_0}(z) = \Phi(z), \tag{22}$$

$$\tilde{\Phi}^{2\Lambda_1,1}_{\Lambda_0+\Lambda_1}(z) = (-q^4z)^{-1/4}\Phi(z), \tag{23}$$

$$\tilde{\Phi}^{2\Lambda_0,1}_{\Lambda_0+\Lambda_1}(z) = (-q^4z)^{1/4}\Phi(z), \tag{24}$$

$$\tilde{\Phi}^{\Lambda_0+\Lambda_1,1}_{2\Lambda_1}(z) = (-q^6z)^{-1/2}\Phi(z). \tag{25}$$

3.2. **type II Vertex Operators for level 2 and spin 1/2.** We consider type II vertex operators of the following kind

$$\Psi^{1,\Lambda_0+\Lambda_1}_{2\Lambda_i}(z) : V(2\Lambda_i) \longrightarrow V^{(1)}_z \otimes V(\Lambda_0+\Lambda_1), \tag{26}$$

$$\Psi^{1,2\Lambda_i}_{\Lambda_0+\Lambda_1}(z) : V(\Lambda_0+\Lambda_1) \longrightarrow V^{(1)}_z \otimes V(2\Lambda_i). \tag{27}$$

Explicit forms of the components are as follows.

$$\Psi_0(z) = B_{II,<}(z)B_{II,>}(z)\Omega(q^{-2}z)e^{-\alpha/2}(-q^2z)^{-\partial/4}, \tag{28}$$

$$\Psi_1(z) = \oint \frac{dw}{2\pi i} B_{II,<}(z)E^+_<(w)B_{II,>}(z)E^+_>(w)\Omega(q^{-2}z)\phi(w) \tag{29}$$

$$\times\, e^{\alpha/2}(-q^2z)^{-\partial/4}w^{\partial/2}(-q^2zw^3)^{-\frac{1}{2}}\frac{(\frac{w}{qz};q^4)_\infty}{(\frac{qw}{z};q^4)_\infty}\Big\{\frac{w}{1-q^{-3}w/z}+\frac{q^3z}{1-qz/w}\Big\},$$

$$B_{II,<}(z) = \exp\big(-\sum_{n=1}^{\infty}\frac{[n]a_{-n}}{[2n]^2}(qz)^n\big), \tag{30}$$

$$B_{II,>}(z) = \exp\big(\sum_{n=1}^{\infty}\frac{[n]a_{n}}{[2n]^2}(q^3z)^{-n}\big). \tag{31}$$

The integrand of $\Psi_1(z)$ has poles only at $w = q^3z, qz$ except for $w = 0, \infty$ and the contour of integration encloses $w = 0, qz$. Subscripts for fermion sectors are abbreviated.

Normalized vertex operators are defined by the conditions

$$\langle\Lambda_0+\Lambda_1|\tilde{\Psi}_1(z)|2\Lambda_0\rangle = 1, \quad \langle 2\Lambda_1|\tilde{\Psi}_1(z)|\Lambda_0+\Lambda_1\rangle = 1,$$
$$\langle\Lambda_0+\Lambda_1|\tilde{\Psi}_0(z)|2\Lambda_1\rangle = 1, \quad \langle 2\Lambda_0|\tilde{\Psi}_0(z)|\Lambda_0+\Lambda_1\rangle = 1,$$

and these are given by

$$\tilde{\Psi}^{1,\Lambda_0+\Lambda_1}_{2\Lambda_0}(z) = (-q)^{-1}\Psi(z), \tag{32}$$

$$\tilde{\Psi}^{1,2\Lambda_1}_{\Lambda_0+\Lambda_1}(z) = -(-q^6z)^{-1/4}\Psi(z), \tag{33}$$

$$\tilde{\Psi}^{1,2\Lambda_0}_{\Lambda_0+\Lambda_1}(z) = (-q^2z)^{1/4}\Psi(z), \tag{34}$$

$$\tilde{\Psi}^{1,\Lambda_0+\Lambda_1}_{2\Lambda_1}(z) = (-q^2z)^{1/2}\Psi(z). \tag{35}$$

3.3. **type II Vertex Operators for level 2 and spin 1.** When the spin of the evaluation module is 1, the type II vertex operators do not contain any fermion emission vertex operators.

$$\Psi^{2,2\Lambda_i}_{2\Lambda_i}(z) : V(2\Lambda_i) \longrightarrow V^{(2)}_z \otimes V(2\Lambda_i), \tag{36}$$

$$\Psi^{2,\Lambda_0+\Lambda_1}_{\Lambda_0+\Lambda_1}(z) : V(\Lambda_0+\Lambda_1) \longrightarrow V^{(2)}_z \otimes V(\Lambda_0+\Lambda_1). \tag{37}$$

Explicit form of the components are as follows.

$$\Psi_0(z) = F_{II,<}(z)F_{II,>}(z)e^{-\alpha}(-q^2z)^{-\partial/2+1}, \tag{38}$$

$$\Psi_1(z) = \oint \frac{dw}{2\pi i} F_{II,<}(z)E^+_<(w)F_{II,>}(z)E^+_>(w)\phi(w)\Big(\frac{w}{-q^2z}\Big)^{\partial/2} \tag{39}$$

$$\times\, w^{-1/2}\Big\{\frac{1}{1-\frac{w}{q^4z}}+\frac{q^4z}{w\big(1-\frac{z}{w}\big)}\Big\},$$

The integration contour encircles poles $w=0,z$ but the pole $w=q^4z$ lies outside of it.

$$\Psi_2(z)=\oint\frac{dw_2}{2\pi i}\oint\frac{dw_1}{2\pi i}F_{II,<}(z)E^+_<(w_1)E^+_<(w_2)F_{II,>}(z)E^+_>(w_1)E^+_>(w_2)$$
$$\times e^{\alpha}\Big(\frac{w_1w_2}{-q^2z}\Big)^{\partial/2}(w_1w_2)^{-1/2}\Big\{\frac{1}{1-\frac{w_1}{q^4z}}+\frac{q^4z}{w_1\big(1-\frac{z}{w_1}\big)}\Big\}$$
$$\times\Big\{[2]^{-1}:\phi(w_1)\phi(w_2):\Bigg(\frac{w_1-q^{-2}w_2}{-q^2z\big(1-\frac{w_2}{q^4w_1}\big)}+\frac{1-\frac{w_1}{q^2w_2}}{1-\frac{z}{w_2}}\Bigg)$$
$$+\frac{(w_1w_2)^{1/2}\big(1-\frac{w_2}{w_1}\big)}{-q^2z\big(1-\frac{q^2w_2}{w_1}\big)\big(1-\frac{w_2}{q^4z}\big)}-\frac{\big(\frac{w_1}{w_2}\big)^{1/2}\big(1-\frac{w_1}{w_2}\big)}{\big(1-\frac{q^2w_1}{w_2}\big)\big(1-\frac{z}{w_2}\big)}\Big\}, \tag{40}$$

We have to prepare two contours because of the fermionic part and one is for the term including $:\phi(w_1)\phi(w_2):$ and the other is for the rest. The former satisfies $|\frac{w_2}{q^4w_1}|<1,|w_2|>|z|$ and the same condition satisfied by the contour for Psi_1 with substitution $w=w_1$. The latter satisfies $|q^2w_2|<|w_1|<|q^{-2}w_2|$ and the same conditions as Psi_1 with $w=w_1,w_2$.

$$F_{II,<}(z)=\exp\Big(-\sum_{m>0}\frac{a_{-m}}{[2m]}(qz)^m\Big), \tag{41}$$

$$F_{II,>}(z)=\exp\Big(\sum_{m>0}\frac{a_m}{[2m]}(q^3z)^{-m}\Big). \tag{42}$$

Under the normailzation

$$\langle 2\Lambda_0|\tilde{\Psi}_0(z)|2\Lambda_1\rangle=1,\quad \langle 2\Lambda_1|\tilde{\Psi}_2(z)|2\Lambda_0\rangle=1,$$
$$\langle\Lambda_0+\Lambda_1|\tilde{\Psi}_1(z)|\Lambda_0+\Lambda_1\rangle=1,$$

$\tilde{\Psi}(z)$' are given by

$$\tilde{\Psi}^{2,2\Lambda_0}_{2\Lambda_1}(z)=\Psi(z), \tag{43}$$

$$\tilde{\Psi}^{2,\Lambda_0+\Lambda_1}_{\Lambda_0+\Lambda_1}(z)=-(-q^2z)^{-1/2}\Psi(z), \tag{44}$$

$$\tilde{\Psi}^{2,2\Lambda_1}_{2\Lambda_0}(z)=(-q^4z)^{-1}\Psi(z). \tag{45}$$

4. Derivation

Taking $\Phi^{\Lambda_0+\Lambda_1,1}_{2\Lambda_i}(z)$ as an example, we discuss the derivation of the results in the previous section. Other cases can be treated in almost the same way.

4.1. **General structure of $\Phi_0(z)$ and $\Phi_1(z)$.** Calculating

$$\Delta(x)\Phi(z)=\Phi(z)x$$

for $x =$ Chevalley generators of U and a_n, we get

$$0 = [\Phi_1(z), x_0^+],$$

$$K\Phi_1(z) = [\Phi_0(z), x_0^+],$$

$$0 = x_0^-\Phi_0(z) - q\Phi_0(z)x_0^-,$$

$$\Phi_0(z) = \Phi_1(z)x_0^- - qx_0^-\Phi_1(z), \tag{46}$$

$$0 = \Phi_0(z)x_1^- - qx_1^-\Phi_0(z),$$

$$q^3z\Phi_0(z) = \Phi_1(z)x_1^- - q^{-1}x_1^-\Phi_1(z), \tag{47}$$

$$(qzK)^{-1}\Phi_1(z) = [\Phi_0(z), x_{-1}^+],$$

$$0 = [\Phi_1(z), x_{-1}^+],$$

$$K\Phi_1(z)K^{-1} = q\Phi_1(z), \tag{48}$$

$$K\Phi_0(z)K^{-1} = q^{-1}\Phi_0(z),$$

$$[a_m, \Phi_1(z)] = (q^5z)^m\frac{[m]}{m}\Phi_1(z), \tag{49}$$

$$[a_{-m}, \Phi_1(z)] = (q^3z)^{-m}\frac{[m]}{m}\Phi_1(z). \tag{50}$$

From (48,49,50), we can speculate the form of $\Phi_1(z)$ as

$$\Phi_1(z) = B_{I,<}(z)B_{I,>}(z)\Omega_{NS}^R(z)e^{\alpha/2}y^{\partial}.$$

To determine y and the fermionic part $\Omega_{NS}^R(z)$, we impose the following conditions on $\Phi_1(z)$

$$\Phi_1(z)x_0^- - qx_0^-\Phi_1(z) = (q^3z)^{-1}(\Phi_1(z)x_1^- - q^{-1}x_1^-\Phi_1(z)),$$

$$0 = [\Phi_1(z), x^+(w)],$$

which can be easily seen from (46,47) and the proposition of Section 4.4 of Ref.[12]. Then we have (10,14)

$$\Phi_1(z) = B_{I,<}(z)B_{I,>}(z)\Omega_{NS}^R(z)e^{\alpha/2}(-q^4z)^{\partial/4},$$

$$\phi^R(w)\Omega_{NS}^R(z) = \Big(\frac{-q^4z}{w}\Big)^{1/2}\frac{\big(\frac{w}{q^3z};q^4\big)_\infty\big(\frac{q^7z}{w};q^4\big)_\infty}{\big(\frac{w}{qz};q^4\big)_\infty\big(\frac{q^5z}{w};q^4\big)_\infty}\Omega_{NS}^R(z)\phi^{NS}(w).$$

$\Phi_1(z)$ can be calculated through (46)

$$\begin{aligned}\Phi_0(z) &= \oint \frac{dw}{2\pi i}\frac{1}{w}\{\Phi_1(z)x^-(w) - qx^-(w)\Phi_1(z)\}\\ &= \oint \frac{dw}{2\pi i} B_{I,<}(z)E^-_<(w)B_{I,>}(z)E^-_>(w)\Omega^R_{NS}(z)\phi^{NS}(w)\\ &\times e^{-\alpha/2}(-q^4z)^{\partial/4}w^{-\partial/2}(-q^4zw^3)^{-\frac{1}{2}}\frac{(\frac{w}{q^3z};q^4)_\infty}{(\frac{w}{qz};q^4)_\infty}\Big\{\frac{w}{1-q^{-3}w/z} + \frac{q^5z}{1-q^5z/w}\Big\},\end{aligned}$$

To determine the contour of integration we have to find the poles of $\Omega^R_{NS}(z)\phi^{NS}(w)$ and this can be seen from

$$\langle R|\Omega^R_{NS}(z)\phi^{NS}(w)|NS\rangle = \frac{(\frac{w}{qz};q^4)_\infty}{(\frac{w}{q^3z};q^4)_\infty}, \quad \langle NS|\Omega^{NS}_R(z)\phi^R(w)|R\rangle = \big(\frac{w}{-q^4z}\big)^{1/2}\frac{(\frac{w}{qz};q^4)_\infty}{(\frac{w}{q^3z};q^4)_\infty}.$$

Hence as a composite $\Omega^R_{NS}(z)\phi^{NS}(w)\frac{(\frac{w}{q^3z};q^4)_\infty}{(\frac{w}{qz};q^4)_\infty}$ in the integrand has no poles and the contour is the one encloses $w = 0, q^5z$.

4.2. **Fermion emission vertex operator.** In Ref.[6], Eqn.(15) appears in the study of the Ising model and its free field realization is given without any details. Thus we give the exposition of its derivation*. The main point of derivating free field realization of the fermion emission vertex operator $\Omega^R_{NS}(z)$ (15,16) is to expand $\Omega^R_{NS}(z)$ as

$$\Omega^R_{NS}(z) = \sum_{K,L} a_{K,L}\ \phi^R_{k_1}\phi^R_{k_2}\cdots|R\rangle\langle NS|\phi^{NS}_{l_1}\phi^{NS}_{l_2}\cdots,$$
$$K = \{k_i\}, L = \{l_i\},$$

and to calculate the coefficients $a_{K,L}$. After normalizing ϕ_n suitably to φ_n (19,20), we see "$a_{K,L}$/(normalization factor)" are identified with Pfaffians of $X_{k,l}$. With the aid of a relation satisfied by Pfaffian

$$\omega^{\wedge n} = n!\mathrm{Pf}(b_{ij})x_1\wedge x_2\cdots\wedge x_{2n},$$

where x_k $(1\le k\le 2n)$ is a Grassmann variable and

$$\omega = \sum_{1\le i<j\le 2n} b_{ij}x_i\wedge x_j,$$

we get (15,16).

*We are indebted to M.Jimbo for explaining the details of Ref.[6].

Wick's theorem can be generalized to the present situation and we only need to calculate one- and two-point correlation functions for $a_{K,L}$. To calculate these, we rewrite (14) and introduce auxiliary operators

$$\tilde{\phi}^{NS}(w)\Omega_R^{NS}(q^{-4}) = \Omega_R^{NS}(q^{-4})\tilde{\phi}^R(w), \tag{51}$$

$$\tilde{\phi}^{NS}(w) = (-1)^{-1/2}w^{1/2}\frac{(qw^{-1};q^4)_\infty}{(q^3w^{-1};q^4)_\infty}\phi^{NS}(w), \tag{52}$$

$$\tilde{\phi}^R(w) = \frac{(qw;q^4)_\infty}{(q^3w;q^4)_\infty}\phi^R(w) = f_+(w)\phi^R(w), \tag{53}$$

we set $\Omega(z=q^4)$ for simplicity. They are defined to satisfy

$$\langle NS|\tilde{\phi}^{NS}_n = 0\ (n<0), \quad \tilde{\phi}^R_n|R\rangle = 0\ (n>0), \quad \tilde{\phi}^R_0|R\rangle = |R\rangle,$$

and this enables us to see that

$$\langle NS|\Omega_R^{NS}(q^{-4})\tilde{\phi}^R(z)\tilde{\phi}^R(w)|NS\rangle = \langle NS|\tilde{\phi}^{NS}(z)\Omega_R^{NS}(q^{-4})\tilde{\phi}^R(w)|NS\rangle$$

contains only negative (positive) powers of z (w). On the other hand the expectation value of

$$\{\tilde{\phi}^R(z),\tilde{\phi}^R(w)\} = f_+(z)f_+(w)\Big(\delta\big(\frac{q^2w}{z}\big)+\delta\big(\frac{w}{q^2z}\big)\Big),$$
$$\delta(z) = \sum_{n\in\mathbb{Z}} z^n,$$

with respect to $\langle NS|\Omega_R^{NS}(q^{-4})$ and $|R\rangle$ is

$$\langle NS|\Omega_R^{NS}(q^{-4})\tilde{\phi}^R(z)\tilde{\phi}^R(w)|NS\rangle + \langle NS|\Omega_R^{NS}(q^{-4})\tilde{\phi}^R(w)\tilde{\phi}^R(z)|NS\rangle$$
$$= f_+(z)f_+(w)\Big(\delta\big(\frac{q^2w}{z}\big)+\delta\big(\frac{w}{q^2z}\big)\Big)$$

where we normalize $\langle NS|\Omega_R^{NS}(q^{-4})|R\rangle = 1$. And we get

$$\langle NS|\Omega_R^{NS}(q^{-4})\tilde{\phi}^R(z)\tilde{\phi}^R(w)|R\rangle = \frac{1-qw}{1-q^2w/z}+\frac{1-q^{-1}w}{1-q^{-2}w/z}-1$$

Expanding the last line of the following equation as in C

$$\langle NS|\Omega_R^{NS}(q^{-4})\phi^R(z)\phi^R(w)|R\rangle = \sum_{n,m\in\mathbb{Z}}\langle NS|\Omega_R^{NS}(q^{-4})\phi^R_n\phi^R_m|R\rangle z^{-n}w^{-m}$$
$$= \frac{1}{f_+(z)f_+(w)}\Big\{\frac{1-qw}{1-q^2w/z}+\frac{1-q^{-1}w}{1-q^{-2}w/z}-1\Big\},$$

we have

$$\langle NS|\Omega_R^{NS}(q^{-4})\phi^R_{-n}\phi^R_{-m}|R\rangle = X_{m,n}\gamma_n\gamma_m q^{n+m}\ (n,m\geq 0). \tag{54}$$

Similar calculation yields

(55) $$\langle NS|\phi^{NS}_{k+1/2}\Omega^{NS}_{R}(q^{-4})\phi^{R}_{-n}|R\rangle = -(-1)^{1/2}X_{-k-1/2,n}\gamma_n\gamma_k q^{n+k}\ (n,k\geq 0),$$

(56) $$\langle NS|\phi^{NS}_{k+1/2}\phi^{NS}_{l+1/2}\Omega^{NS}_{R}(q^{-4})|R\rangle = -X_{l+1/2,k+1/2}\gamma_l\gamma_k q^{l+k}\ (k,l\geq 0).$$

z-dependence of $\Omega^{R}_{NS}(z)$ is recovered with the equation

(57) $$\begin{aligned}\zeta^{d^R}\Omega^{R}_{NS}(z)\zeta^{-d^{NS}} &= \Omega^{R}_{NS}(\zeta^{-1}z),\\ \zeta^{-d^i}\phi^i(z)\zeta^{d^i} &= \phi^i(\zeta z),\\ \langle i|d^i = d^i|i\rangle &= 0,\end{aligned}$$

where d^i's are the fermionic part of d of (3)

$$d^i = -\sum_{k>0} kN^{\phi^i}_k,\ (i = NS \text{ or } R)$$

and satisfy

$$[d^i,\phi^i_n] = n\phi_n.$$

To derive (57), we multiply (14) by $\zeta^{d^R},\zeta^{-d^{NS}}$ from left and right respectively.

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Appendix A. boson

Followings are useful formulae for normal ordering bosons. We set $(z)_\infty = (z;q^4)_\infty$ for brevity.

$$\begin{aligned}B_{I,>}(z)E^-_<(w) &= \frac{(qw/z)_\infty}{(q^{-1}w/z)_\infty}E^-_<(w)B_{I,>}(z),\\ E^-_>(w)B_{I,<}(z) &= \frac{(q^9z/w)_\infty}{(q^7z/w)_\infty}B_{I,<}(z)E^-_>(w),\\ B_{I,>}(z)E^+_<(w) &= \frac{(q^{-3}w/z)_\infty}{(q^{-1}w/z)_\infty}E^+_<(w)B_{I,>}(z),\\ E^+_>(w)B_{I,<}(z) &= \frac{(q^5z/w)_\infty}{(q^7z/w)_\infty}B_{I,<}(z)E^+_>(w),\\ B_{II,>}(z)E^+_<(w) &= \frac{(q^{-1}w/z)_\infty}{(q^{-3}w/z)_\infty}E^+_<(w)B_{II,>}(z),\\ E^+_>(w)B_{II,<}(z) &= \frac{(q^3z/w)_\infty}{(qz/w)_\infty}B_{II,<}(z)E^+_>(w),\end{aligned}$$

$$B_{II,>}(z)E^-_<(w)=\frac{(q^{-1}w/z)_\infty}{(qw/z)_\infty}E^-_<(w)B_{II,>}(z),$$
$$E^-_>(w)B_{II,<}(z)=\frac{(q^3z/w)_\infty}{(q^5z/w)_\infty}B_{II,<}(z)E^-_>(w),$$
$$F_{II,>}(z)E^-_<(w)=(1-\frac{w}{q^2z})E^-_<(w)F_{II,>}(z),$$
$$E^-_>(w)F_{II,<}(z)=(1-\frac{q^2z}{w})F_{II,<}(z)E^-_>(w),$$
$$F_{II,>}(z)E^+_<(w)=\frac{1}{1-q^{-4}w/z}E^+_<(w)F_{II,>}(z),$$
$$E^+_>(w)F_{II,<}(z)=\frac{1}{1-z/w}F_{II,<}(z)E^-_>(w),$$
$$E^-_>(w_1)E^+_<(w_2)=\frac{1}{1-w_2/w_1}E^+_<(w_2)E^-_>(w_1),$$
$$E^+_>(w_2)E^-_<(w_1)=\frac{1}{1-w_1/w_2}E^-_<(w_1)E^+_>(w_2),$$

Appendix B. fermion

For $\Omega^{NS}_R(z)$, we show the equations corresponding to the ones from (51) to (56)

(58) $$\tilde\phi^{R'}(w)\Omega^R_{NS}(q^{-4})=\Omega^R_{NS}(q^{-4})\tilde\phi^{NS'}(w),$$

(59) $$\tilde\phi^{R'}(w)=\frac{(q/w;q^4)_\infty}{(q^3/w;q^4)_\infty}\phi^R(w),$$

(60) $$\tilde\phi^{NS'}(w)=(-1)^{1/2}w^{-1/2}\frac{(qw;q^4)_\infty}{(q^3w;q^4)_\infty}\phi^{NS}(w),$$

$$\langle R|\tilde\phi^{R'}_n=0\ (n<0),\quad \langle R|\tilde\phi^{R'}_0=\langle R|,\quad \tilde\phi^{NS'}_n|NS\rangle=0\ (n>0),$$

$$\langle R|\tilde\phi^{R'}(z)\tilde\phi^{R'}(w)\Omega^R_{NS}(q^{-4})|NS\rangle=\frac{1-q/z}{1-q^2w/z}+\frac{1-q^{-1}/z}{1-q^{-2}w/z}-1$$

$$\langle R|\phi^R_n\phi^R_m\Omega^R_{NS}(q^{-4})|NS\rangle=X_{n,m}\gamma_n\gamma_m q^{n+m}\ (n,m\geq 0),$$
$$\langle R|\phi^R_n\Omega^R_{NS}(q^{-4})\phi^{NS}_{-k-1/2}|NS\rangle=(-1)^{1/2}X_{-k-1/2,n}\gamma_n\gamma_k q^{n+k}\ (n,k\geq 0),$$
$$\langle R|\Omega^R_{NS}(q^{-4})\phi^{NS}_{-k-1/2}\phi^{NS}_{-l-1/2}|NS\rangle=X_{l+1/2,k+1/2}\gamma_l\gamma_k q^{l+k}\ (k,l\geq 0)$$

Appendix C. Calculation of Eqn.(54)

We show details of calculation of (54). From (21)

$$
\begin{aligned}
&\langle NS|\Omega_R^{NS}(q^{-4})\phi^R(z)\phi^R(w)|R\rangle \\
&= \frac{1}{f_+(z)f_+(w)}\Big\{\frac{1-qw}{1-q^2w/z}+\frac{1-q^{-1}w}{1-q^{-2}w/z}-1\Big\} \\
&= \sum_{k\geqslant 0, l\geqslant 0}\gamma_k(qz)^k\gamma_l(qw)^l\Big\{\sum_{a\geqslant 0}\Big((1-qw)\big(\frac{q^2w}{z}\big)^a+(1-w/q)\big(\frac{w}{q^2z}\big)^a\Big)-1\Big\} \\
&= \sum_{0\leqslant a\leqslant m}\gamma_{n+a}\gamma_{m-a}\eta_a q^{n+m}z^n w^m-\sum_{0\leqslant a\leqslant m-1}\gamma_{n+a}\gamma_{m-a-1}(q^{2a}+q^{-2(a+1)})q^{n+m}z^nw^m-\gamma_n\gamma_m z^n w^m
\end{aligned}
$$

Hence the equation to be proved is

$$
X_{n,m}\gamma_n\gamma_m=\sum_{0\leqslant a\leqslant m}\gamma_n\gamma_m\eta_a-\sum_{0\leqslant a\leqslant m-1}\gamma_{n+a}\gamma_{m-a-1}(q^{2a}+q^{-2(a+1)})-\gamma_n\gamma_m z^n w^m,
$$

which is equivalent to

$$
X_{n,m}=1+(1-t^{-1})(1+t^{2n})\sum_{1\leqslant a\leqslant m}\frac{(t^{1+2n};t^2)_{a-1}}{(t^{2+2n};t^2)_{a-1}}\frac{(t^{2m-2a+2};t^2)_a}{(t^{2m-2a+1};t^2)_a}\frac{t^a}{1-t^{2(n+a)}} \tag{61}
$$

where we set $t=q^2$. It can be proved by induction with respect to k that the summation over $a=m,m-1,\cdots,m-k$ yields

$$
t^{m-k}\frac{(t^{1+2n};t^2)_{m-k-1}}{(t^{2+2n};t^2)_{m-k-1}}\frac{(t^{2k+2};t^2)_{m-k}}{(t^{2k+1};t^2)_{m-k}}\frac{\sum_{j=0}^k t^{2j}}{1-t^{2(n+k)}}.
$$

Setting $k=m-1$ we can see that the right hand side of (61) is equal to $\frac{t^{2m}-t^{2n}}{1-t^{2(n+m)}}$.

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